Math 172

Review for the final exam.

1. Find the integrals.(8 points each)

   (a)
$$\int x \arctan x \, dx;$$

   Solution. We perform integration by parts taking $u = \arctan x$ and $dv = xdx$. Then
$$du = \frac{1}{1+x^2}dx \text{ and } v = \frac{x^2}{2}.$$

   Therefore
$$\int x \arctan x \, dx = \frac{x^2}{2} \arctan x - \int \frac{x^2}{2(1+x^2)} \, dx =$$
$$= \frac{x^2}{2} \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) = \frac{x^2}{2} \arctan x - \frac{x}{2} + \frac{1}{2} \arctan x + C =$$
$$= \frac{x^2+1}{2} \arctan x - \frac{x}{2} + C.$$

   (b)
$$\int \frac{1}{t^2+2t+2} \, dt;$$

   Solution. By completing the square we get
$$\int \frac{1}{t^2+2t+2} \, dt = \int \frac{1}{(t+1)^2+1} \, dt.$$

   Let $u = t+1$ then $du = dt$ and
$$\int \frac{1}{(t+1)^2+1} \, dt = \int \frac{1}{u^2+1} \, du = \arctan u + C = \arctan(t+1) + C.$$

(c)

$$\int \cos^3 x \sin^{\frac{1}{2}} x \, dx;$$

Solution. First we write this integral in the form

$$\int \cos^2 x \sin^{\frac{1}{2}} x \cos x \, dx = \int (1 - \sin^2 x) \sin^{\frac{1}{2}} x \cos x \, dx$$

and then make the substitution $u = \sin x$, whence $du = \cos x dx$. Thus our integral becomes

$$\int (1 - \sin^2 x) \sin^{\frac{1}{2}} x \cos x \, dx = \int (1 - u^2) u^{\frac{1}{2}} \, du =$$

$$= \int u^{\frac{1}{2}} - u^{\frac{5}{2}} \, du = \frac{2}{3} u^{\frac{3}{2}} - \frac{2}{7} u^{\frac{7}{2}} + C =$$

$$= \frac{2}{3} \sin^{\frac{3}{2}} x - \frac{2}{7} \sin^{\frac{7}{2}} x + C.$$

(d)

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \, dx.$$

Solution. Probably the simplest way to solve the problem is to notice that

$$\frac{d}{dx}(e^{2x} - e^{-2x}) = 2e^{2x} - (-2)e^{-2x} = 2(e^{2x} + e^{-2x}).$$

Thus, the numerator of the fraction we integrate equals to one half of the derivative of the denominator whence

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \, dx = \frac{1}{2} \ln(e^{2x} + e^{-2x}) + C.$$

Remark The answer also can be written in the form

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} \, dx = \frac{1}{2} \cosh 2x + C.$$

2. Find the value of the improper integral (10 points)

$$\int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx.$$

Solution. We perform the substitution $x = \sec t$. Then $dx = \sec t \tan t dt$ and $\sqrt{x^2-1} = \sqrt{\sec^2 t - 1} = \sqrt{\tan^2 t} = \tan t$. We have to find out how the limits of integration will change under the substitution. If $x = 2$ then $\sec t = 2$ whence $\cos t = \frac{1}{2}$ and $t = \arccos\frac{1}{2} = \frac{\pi}{3}$. If $x \to \infty$ then $\cos t = \frac{1}{\sec t} = \frac{1}{x}$ whence $t = \arccos\frac{1}{x} \to \arccos 0 = \frac{\pi}{2}$. Therefore after the substitution we get a **proper** integral

$$\int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sec t \tan t}{\sec t \tan t}\,dt = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} dt = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.$$

3. Find the volume of the solid generated when the region bounded by $y = 2-x, y = \sqrt{x}$ and $x = 0$ is revolved about x-axis.(10 points)
Solution. The graph of the region is shown below. (On the test you do not have to provide the graph) To find the point of intersection of the curves $y = 2-x$ and $y = \sqrt{x}$ we have to solve the equation $2 - x = \sqrt{x}$. After we square both parts we get $4 - 4x + x^2 = x$ or $x^2 - 5x + 4 = (x-1)(x-4) = 0$. Only $x = 1$ is the solution of our original equation.

Next we apply the disks' method to find the volume.

$$V_x = \pi\int_0^1 [(2-x)^2 - (\sqrt{x})^2]\,dx = \pi\int_0^1 (x^2 - 5x + 4)\,dx =$$

$$= \pi\Big(\frac{x^3}{3} - \frac{5x^2}{2} + 4x\Big)\Big|_0^1 = \pi\Big(\frac{1}{3} - \frac{5}{2} + 4\Big) = \frac{11}{6}\pi.$$

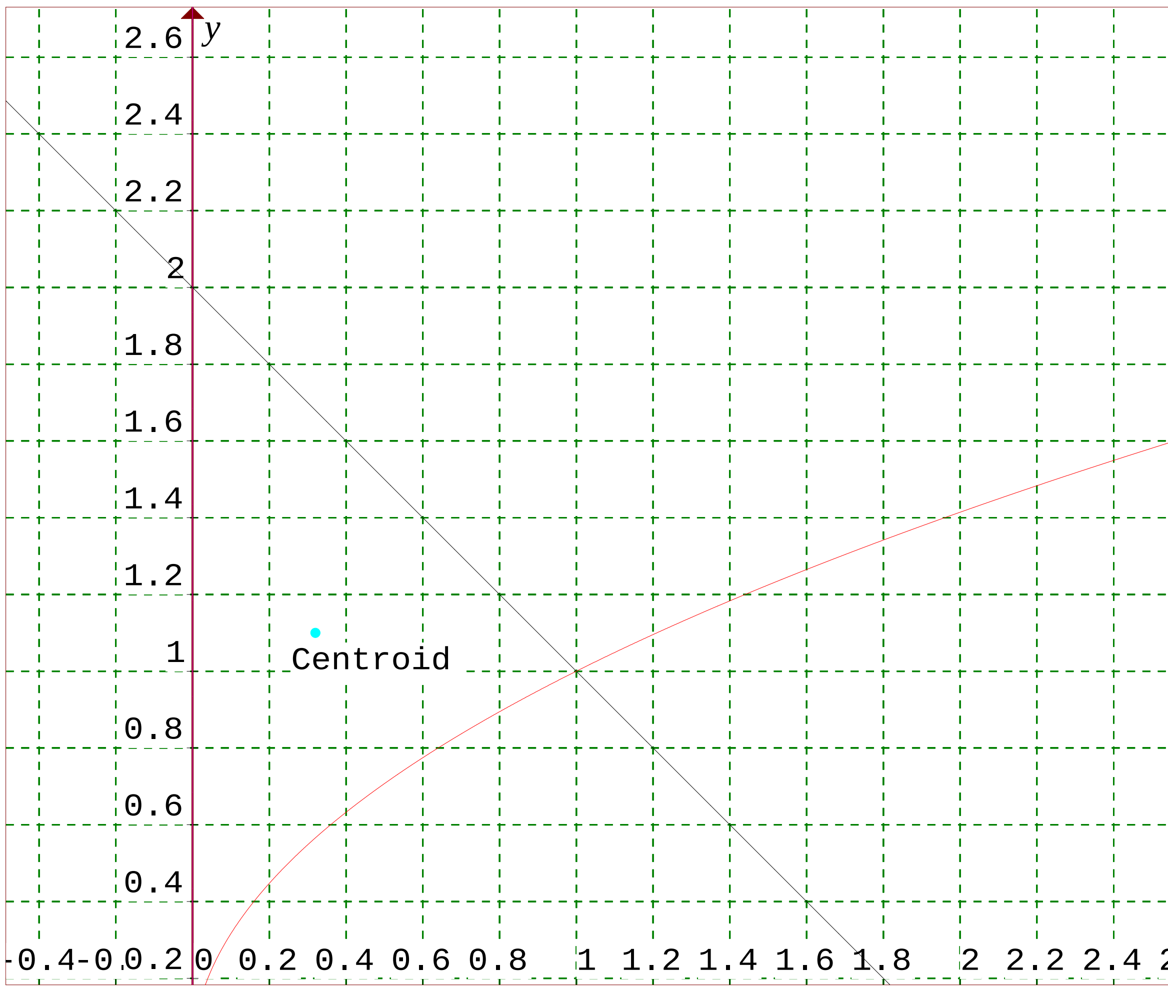
4. Find the volume of the solid generated when the region bounded by $y = 2-x, y = \sqrt{x}$ and $x = 0$ is revolved about y-axis.(10 points)

Solution. We apply the cylindrical shells' method.

$$V_y = 2\pi \int\limits_0^1 x[(2-x) - \sqrt{x}]\,dx = 2\pi \int\limits_0^1 (2x - x^2 - x^{\frac{3}{2}})\,dx =$$

$$= 2\pi\left(x^2 - \frac{x^3}{3} - \frac{2}{5}x^{\frac{5}{2}}\right)\Bigg|_0^1 = \frac{8\pi}{15}.$$

Graphmatica 2.0e © 2005 kSoft, Inc. - region.gr



Equations on screen:
1. y=sqrt(x)
2. y=2-x
3. x=0

Data Plots:
Data plot 1

5. Find the coordinates of the centroid of the region described in the previous problem. (10 points) **Solution.** First we have to find the area of the region.

$$A = \int_0^1 (2 - x - \sqrt{x})\, dx = \left(2x - \frac{x^2}{2} - \frac{2}{3}x^{\frac{3}{2}}\right)\Big|_0^1 = 2 - \frac{1}{2} - \frac{2}{3} = \frac{5}{6}.$$

Now we can compute the coordinates of centroid.

$$x_c = \frac{V_y}{2\pi A} = \frac{8\pi}{15} \div \frac{5\pi}{3} = \frac{8}{25} = 0.32,$$

$$y_c = \frac{V_x}{2\pi A} = \frac{11\pi}{6} \div \frac{5\pi}{3} = \frac{11}{10} = 1.1.$$

6. Determine whether the series absolutely converges, conditionally converges, or diverges.(5 points each)

(a)

$$\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k^6 - 3k}};$$

Solution. We apply the limit comparison test. By keeping only the leading terms in the numerator and the denominator we get the following series

$$\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k^6}} = \sum_{k=2}^{\infty} \frac{1}{k^2}.$$

The last series converges because the exponent of k in the denominator is greater than 1. By limit comparison test our original series also converges.

(b)

$$\sum_{k=1}^{\infty} (-1)^k \frac{k}{k^2 + 2};$$

Solution. This is an alternating series. Indeed,

- the sign of the terms alternates;

- $$\lim_{k\to\infty} \frac{k}{k^2+2} = 0;$$
- The absolute values of the terms of the series are decreasing. To prove it it is enough to prove that the function $g(x) = \frac{x}{x^2+2}$ is decreasing. Indeed, by the quotient rule

  $$\frac{dg}{dx} = \frac{x^2+2-x(2x)}{(x^2+2)^2} = \frac{2-x^2}{(x^2+2)^2}.$$

  The last expression is negative if $x > \sqrt{2}$ and therefore the absolute values of the terms of the series are decreasing starting from $k = 2$.

So the series converges as an alternating series. To see whether it converges absolutely or conditionally let us look at the series of absolute values,

$$\sum_{k=1}^{\infty} \frac{k}{k^2+2}.$$

The limit comparison test tells us that this series converges or diverges at the same time as the series

$$\sum_{k=1}^{\infty} \frac{k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k}.$$

But the last series diverges and therefore the series of absolute values diverges and our original series converges conditionally.

(c)

$$\sum_{k=1}^{\infty} \frac{(n!)^2}{(2n)!};$$

Solution. We will apply the ratio test.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!^2}{[2(n+1)]!} \div \frac{(n!)^2}{(2n)!} = \left(\frac{(n+1)!}{n!}\right)^2 \frac{(2n)!}{(2n+2)!} =$$

$$= \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{n+1}{2(2n+1)} \underset{n\to\infty}{\to} \frac{1}{4} < 1.$$

By ratio test the series converges.

7. Find the function to which the series

$$\sum_{k=1}^{\infty}(k-1)x^{k+1}$$

converges.(10 points)

Solution. The solution is based on two formulas. The first one is the infinite geometric progression.

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \ -1 < x < 1. \tag{$\star$}$$

The second we obtain if we differentiate both parts of ($\star$).

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}, \ -1 < x < 1. \tag{$\star\star$}$$

Now we can compute the sum in our problem.

$$\sum_{k=1}^{\infty}(k-1)x^{k+1} = \sum_{k=1}^{\infty} kx^{k+1} - \sum_{k=1}^{\infty} x^{k+1} =$$

$$= x^2\sum_{k=1}^{\infty} kx^{k-1} - x^2\sum_{k=1}^{\infty} x^{k-1} =$$

(where $i = k-1$))

$$= x^2\sum_{k=1}^{\infty} kx^{k-1} - x^2\sum_{i=0}^{\infty} x^{i} =$$

(by ($\star$) and ($\star\star$))

$$= \frac{x^2}{(1-x)^2} - \frac{x^2}{1-x} = \frac{x^2 - x^2(1-x)}{(1-x)^2} = \frac{x^3}{(1-x)^2}.$$

Of course, the series converges and the formula is correct only on the interval $-1 < x < 1$.

8. Find the Taylor series about $x = a$ for the given function; express your answer in sigma notation (Σ); then find its radius of convergence and the interval of convergence.(10 points each)

(a)

$$f(x) = \frac{1}{2+x},$$

at 0.

Solution. Plugging in $-x$ instead of x into formula $(\star)$ we get

$$\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k, \ -1 < x < 1. \qquad (\star\star\star)$$

Therefore

$$\frac{1}{2+x} = \frac{1}{2(x+\frac{1}{2}} = \frac{1}{2}\frac{1}{1+\frac{x}{2}} = \frac{1}{2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^k =$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{x^k}{2^{k+1}}, \ -1 < \frac{x}{2} < 1, \text{ or } -2 < x < 2.$$

The radius of convergence of the Maclaurin series above is 2 and the interval of convergence is $(-2, 2)$.

(b)

$$f(x) = \ln x,$$

at 2.

Solution. It would be not very difficult to use the formula for Taylor series of function f at point a

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k, \text{ where} f^{(0)} = f,$$

but it is easier to reduce the problem to one of our standard Maclaurin series. Recall that integrating both parts in $(\star\star\star)$ we get

$$\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}, \ -1 < x < 1. \qquad (\star\star\star\star)$$

Let $u = x - 2$ then $x = u + 2$ and, using $(\star\star\star\star)$ we obtain

$$\ln x = \ln(u+2) = \ln[2(1+\frac{u}{2})] = \ln 2 + \ln(1+\frac{u}{2}) = \ln 2 + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(u/2)^k}{k} =$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{u^k}{2^k k} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-2)^k}{2^k k}.$$

The series above converges if $-1 < u/2 < 1$, or $-2 < u < 2$, or $-2 < x - 2 < 2$, or $0 < x < 4$. Therefore its radius of convergence is 2 and its interval of convergence is $(0, 4)$.