Calculus 1 171 Review 3

In Problems (1) – (4) consider the function $f(x) = (x^2 + x)e^x$.

1. Find the critical (stationary) points; establish their character (relative minimum, relative maximum, or neither); find intervals where the function is increasing or decreasing.

Solution. By product rule we have $f'(x) = (2x+1)e^x + (x^2 + x)e^x = (x^2 + 3x + 1)e^x$. The stationary points will be the solutions of quadratic equation $x^2 + 3x + 1 = 0$. By the quadratic formula $x = \frac{-3 \pm \sqrt{3^2 - 4\times1\times1}}{2\times1} = \frac{-3 \pm \sqrt{5}}{2}$. These two stationary points divide the real axis into three intervals. The next table shows the behavior of the function *f* on these intervals

Interval	$\left(-\infty, \frac{-3-\sqrt{5}}{2}\right)$	$\left(\frac{-3-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}\right)$	$\left(\frac{-3+\sqrt{5}}{2}, \infty\right)$
Sign of $f'(x)$	+	-	+
Behavior of $f(x)$	**Increasing** ↑	**Decreasing** ↓	**Increasing** ↑

Therefore the point $\frac{-3-\sqrt{5}}{2}$ is a relative maximum and the point $\frac{-3+\sqrt{5}}{2}$ is a relative minimum.

2. Find the inflection points and the intervals where the function is concave up or concave down.

Solution. Applying again the product rule we see that $f''(x) = (2x+3)e^x + (x^2 + 3x + 1)e^x = (x^2 + 5x + 4)e^x = (x+1)(x+4)e^x$. There are two inflection points at -1 and at -4. The table below shows the intervals of concavity up and down.

Interval	$(-\infty, -4)$	$(-4, -1)$	$(-1, \infty)$
Sign of $f''(x)$	+	-	+
Concavity	**Up** ∪	**Down** ∩	**Up** ∪

3. Graph the function (your graph should show all the essential details (The x- intercepts, critical points, and inflection points).

To find the *x*-intercepts we have to solve the equation

$f(x) = (x^2 + x)e^x = x(x+1)e^x = 0$ whence the *x*-intercepts are at $x = -1$ and $x = 0$.

Moreover we see that the function is positive on $(-\infty,-1) \cup (0,\infty)$ and negative on $(-1,0)$. A computer generated graph is shown below.

4. Use Maclaurin polynomial of degree 3 to estimate the value of $f(0.1)$. Estimate the error of your approximation.

Solution. Recall the formula for Maclaurin polynomial of degree 3;

$M_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$**.** We already know that

$f'(x) = (x^2 + 3x + 1)e^x$ and $f''(x) = (x^2 + 5x + 4)e^x$. Therefore by the product rule

$f'''(x) = (2x+5)e^x + (x^2 + 5x + 4)e^x = (x^2 + 7x + 9)e^x$ and we have

$M_3(x) = x + 2x^2 + \frac{3}{2}x^3$. Therefore $M_3(0.1) = 0.1 + 0.02 + 0.0015 = 0.1215$.

Recall that the absolute value of the error of our approximation is not greater than $\max\limits_{0 \le t \le 0.1} \frac{|f^{(4)}(t)|}{4!} 0.1^4$. The fourth derivative of *f* is

$f^{(4)}(x) = (2x+7)e^x + (x^2 + 7x + 9)e^x = (x^2 + 9x + 16)e^x$.

The function $f^{(4)}(x)$ obviously is positive and increasing on the interval $[0, 0.1]$. Therefore the error is not greater than $\max\limits_{0\le t\le 0.1}\dfrac{|f^{(4)}(t)|}{4!}0.1^4 = \dfrac{0.1^2+9\times 0.1+16}{24}e^{0.1}\times 0.1^4 \approx .00007786850093$.

For comparison, we can compute directly that $f(0.1)\approx .1215688010$ and therefore the error of our approximation is about .0000688010 which is smaller than our estimate above.

---

5. Find the limit $\lim\limits_{x\to\pi/2}(1+\cos x)^{\tan x}$.

Solution. Because $\cos\dfrac{\pi}{2}=0,\ \lim\limits_{x\to\pi/2-}\tan x=\infty,$ and $\lim\limits_{x\to\pi/2+}\tan x=-\infty$ we have here an indeterminate form 1^∞. As always with this kind of problem we will try first to find the limit of natural logarithm of our expression.
$\ln\left[(1+\cos x)^{\tan x}\right]=\tan x\ln(1+\cos x)$**.** The limit $\lim\limits_{x\to\pi/2}\tan x\ln(1+\cos x)$ is an indeterminate form $0\cdot\infty$. We will rewrite it as $\lim\limits_{x\to\pi/2}\dfrac{\ln(1+\cos x)}{\cot x}$ which is an indeterminate form $\dfrac{0}{0}$ and now we can apply the L'Hospital's rule. According to this rule we differentiate the numerator and the denominator and look at the limit
$\lim\limits_{x\to\pi/2}\dfrac{1/(1+\cos x)(-\sin x)}{-\csc^2 x}=1.$

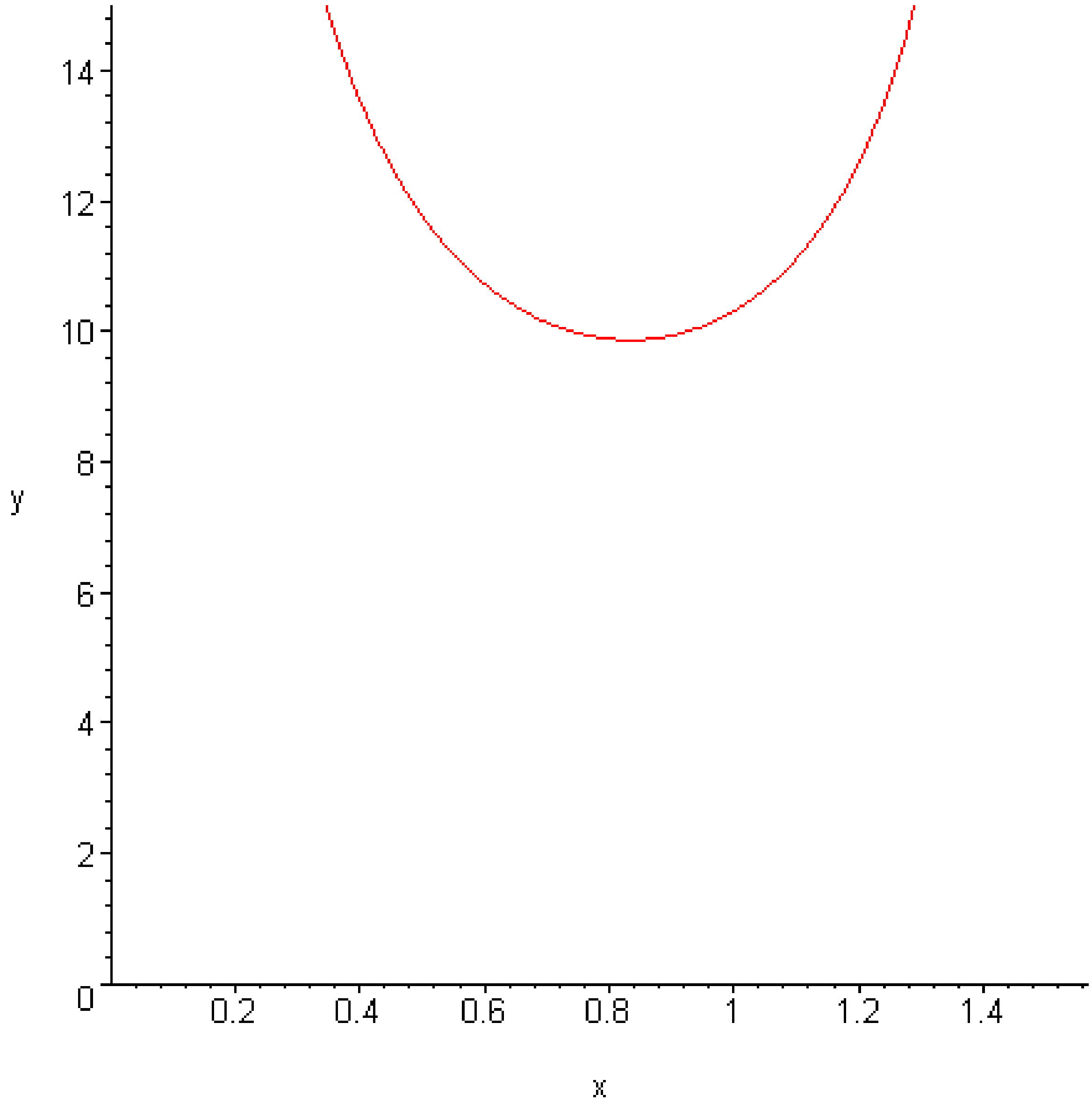
Finally, $\lim\limits_{x\to\pi/2}(1+\cos x)^{\tan x}=e^1=e$.

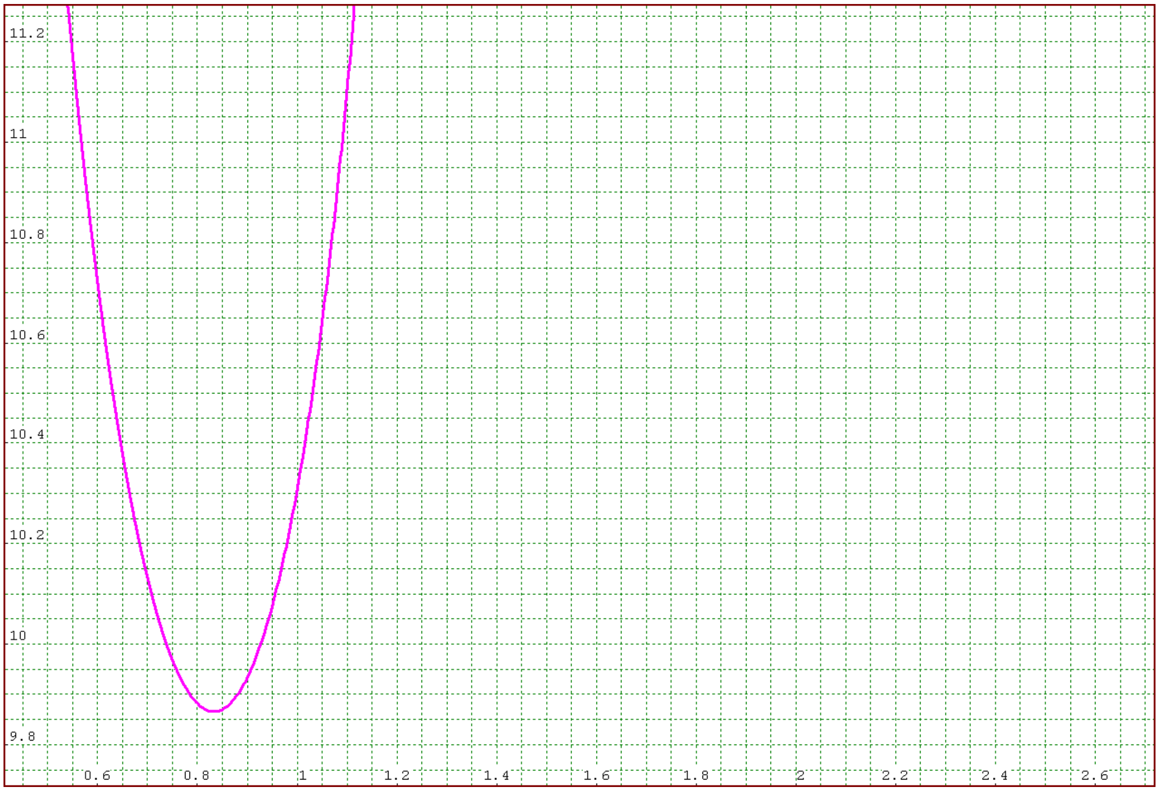
6. Find the smallest value of the function $f(x)=3\sec x+4\csc x$ on the interval $(0,\pi/2)$.

Solution. Because the function is differentiable on $(0,\pi/2)$ it takes its smallest value at a stationary point. Thus we have to solve the equation $\dfrac{df}{dx}=0$.

$\dfrac{df}{dx}=3\tan x\sec x-4\cot x\csc x=\dfrac{3\sin x}{\cos^2 x}-\dfrac{4\cos x}{\sin^2 x}=\dfrac{3\sin^3 x-4\cos^3 x}{\sin^2 x\cos^2 x}$**,** and therefore at a stationary point we have $3\sin^3 x=4\cos^3 x$ whence
$\tan^3 x=4/3$ and $x=\arctan(\sqrt[3]{4/3})\approx 0.83$.
To find the smallest value itself let us notice that at the stationary point $\sec x=\sqrt{1+\tan^2 x}=\sqrt{1+\sqrt[3]{16/9}}$ and $\csc x=\sqrt{1+\cot^2 x}=\sqrt{1+\sqrt[3]{9/16}}$ whence
$\min\limits_{(0,\pi/2)} f(x)=3\sqrt{1+\sqrt[3]{16/9}}+4\sqrt{1+\sqrt[3]{9/16}}\approx 9.87$.

The two computer generated graphs below (made with “MAPLE” and “GRAPHMATICA”, respectively) are included to illustrate our calculations. On the test you are not required to graph the function in such a problem.





7. Use Newton's method to approximate the positive solution of the equation $P(x) = x^3 + x^2 - 1 = 0$.

Solution. Because the sign of the coefficients of the polynomial P changes only once the equation has only one positive real root by the Descartes' rule of signs. Because $P(0) = -1$ and $P(1) = 1$ this root is located between 0 and 1 and we will take as the initial approximation $x_0 = 0.5$. Now it remains to use successively the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n^2 - 1}{3x_n^2 + 2x_n}$. Below it is shown how the calculations can be organized for TI calculators (TI-83, TI-85, et cetera). If you have a different type of calculator you have to organize your calculations yourself or consult the manual.

$0.5 \xrightarrow{\text{STO}} X$ (This command gives x the initial value of 0.5)

$X - (X \wedge 3 + X^2 - 1)/(3X^2 + 2X) \xrightarrow{\text{STO}} X$ (This command computes x_1)

[2nd] [Enter] [Enter] (This command repeats the previous one and therefore computes x_2).

We repeat the last command until the desired accuracy is achieved (e.g. two successive approximations coincide up to the accuracy of our calculator).

Below is shown how it looks if we use TI-85.

$x_0 = 0.5$

$x_1 = 0.857142857143$

$x_2 = 0.764136904762$

$x_3 = 0.754963482395$

$x_4 = 0.754877673714$

$x_5 = 0.754877666247$

$x_6 = 0.754877666247$

Here we stop, the accuracy of our approximation is at least 10^{-11} or one over one hundred billion.

8. An open cup is in the shape of a right circular cone and must have the volume of V cm^3. Find the radius and the height of the cone that would minimize its surface area and also find the minimum value of the surface area.

Solution. The volume and the surface area of a right circular cone with radius r and height h are given by the formulas

$V = \frac{1}{3}\pi r^2 h$

$S = \pi r\sqrt{r^2 + h^2}$

Minimizing S is equivalent to minimizing S^2. Therefore our problem can be written as follows.

Minimize $r^2(r^2 + h^2)$ under the condition that $r^2 h = \frac{3V}{\pi}$. We will use implicit differentiation assuming that r is a function of h. Differentiating both parts of the relation $r^2 h = \frac{3V}{\pi}$ by r we obtain

$$2rh + r^2\frac{dh}{dr} = 0$$

Whence

$$\frac{dh}{dr} = -2\frac{h}{r}$$

The derivative of the function $r^2(r^2 + h^2) = r^4 + r^2h^2$ by r is

$$4r^3 + 2rh^2 + 2r^2h\frac{dh}{dr}$$

At a critical point we have

$$4r^3 + 2rh^2 + 2r^2 h\frac{dh}{dr} = 0$$

Whence

$$2r^2 + h^2 + rh\frac{dh}{dr} = 0$$

By plugging into the last equation $\frac{dh}{dr} = -2\frac{h}{r}$ we get $2r^2 = h^2$ or

$$h = \sqrt{2}r$$

From the last formula we obtain $r^3 = \frac{3V}{\pi\sqrt{2}}$, and $r = \frac{\sqrt[3]{3}}{\sqrt[3]{\pi}\sqrt[6]{2}}\sqrt[3]{V}$.

Then $h = \sqrt{2}r = \sqrt[3]{\frac{6}{\pi}}\sqrt[3]{v}$ and $S = \pi\sqrt{3}r^2 = \pi^{2/3}3^{7/6}2^{-1/6}V^{3/2}$.