Find the following limits (if the limit is positive or negative infinity, or does not exist state it explicitly).

1 $\lim\limits_{x\to 4}\frac{x^2-16}{x^3-64}$. This is an indeterminate form 0/0. To solve the problem we factor the numerator as difference of two squares
$x^2-16=x^2-4^2=(x+4)(x-4)$ and the denominator as difference of two cubes (according to the formula $a^3-b^3=(a-b)(a^2+ab+b^2)$),
$x^3-64=x^3-4^3=(x-4)(x^2+4x+16)$.
Therefore, $\lim\limits_{x\to 4}\frac{x^2-16}{x^3-64}=\lim\limits_{x\to 4}\frac{(x-4)(x+4)}{(x-4)(x^2+4x+16)}=\lim\limits_{x\to 4}\frac{x+4}{x^2+4x+16}=\frac{8}{48}=\frac{1}{6}$.

2. $\lim\limits_{x\to\infty}(\sqrt{x^2-3x+1}-\sqrt{x^2+3x+1})$. Here we have an indeterminate form $\infty-\infty$.

To solve the problem we will first convert it to an indeterminate form $\frac{\infty}{\infty}$ in the following way.

$$\lim_{x\to\infty}(\sqrt{x^2-3x+1}-\sqrt{x^2+3x+1})=\lim_{x\to\infty}\frac{\left(\sqrt{x^2-3x+1}-\sqrt{x^2+3x+1}\right)\left(\sqrt{x^2-3x+1}+\sqrt{x^2+3x+1}\right)}{\sqrt{x^2-3x+1}+\sqrt{x^2+3x+1}}=$$

$$=\lim_{x\to\infty}\frac{(x^2-3x+1)-(x^2+3x+1)}{\sqrt{x^2-3x+1}+\sqrt{x^2+3x+1}}=\lim_{x\to\infty}\frac{-6x}{\sqrt{x^2-3x+1}+\sqrt{x^2+3x+1}}.$$

Because it is now an indeterminate form $\frac{\infty}{\infty}$ the value of the limit will not change if we leave only the leading terms in the denominator, i.e.

$$\lim_{x\to\infty}\frac{-6x}{\sqrt{x^2-3x+1}+\sqrt{x^2+3x+1}}=\lim_{x\to\infty}\frac{-6x}{\sqrt{x^2}+\sqrt{x^2}}=\lim_{x\to\infty}\frac{-6x}{2x}=-3.$$

3. $\lim\limits_{x\to 0}\frac{2^{3x}-1}{3^{2x}-1}$. Again we have an indeterminate form 0/0. We will solve the problem with the help of one of our basic exponential limits $\lim\limits_{u\to 0}\frac{e^u-1}{u}=1$. To do it notice that $2=e^{\ln 2}$ and $3=e^{\ln 3}$ whence $2^{3x}=e^{(3\ln 2)x}$ and $3^{2x}=e^{(2\ln 3)x}$. Next,

$\lim_{x\to 0}\frac{2^{3x}-1}{3^{2x}-1}=\lim_{x\to 0}\frac{e^{(3\ln 2)x}-1}{e^{(2\ln 3)x}-1}=\lim_{x\to 0}\frac{e^{(3\ln 2)x}-1}{3\ln 2x}\times\frac{2\ln 3x}{e^{(2\ln 3)x}-1}\times\frac{3\ln 2}{2\ln 3}$. The limits of the first and the second factor are equal to one (put in the first case $u=3\ln 2x$ and in the second $u=2\ln 3x$ and apply the basic limit $\lim_{u\to 0}\frac{e^u-1}{u}=1$). Therefore the answer in this problem is $\frac{3\ln 2}{2\ln 3}$.

4. $\lim_{s\to 0}\frac{\cos(3s)-1}{\sin^2(5s)}$**.** We deal with this indeterminate form 0/0 in the following way

$$\lim_{s\to 0}\frac{\cos(3s)-1}{\sin^2(5s)}=\lim_{s\to 0}\frac{[\cos(3s)-1][\cos(3s)+1]}{\sin^2(5s)[\cos(3s)+1]}=\lim_{s\to 0}\frac{\cos^2(3s)-1}{\sin^2(5s)[\cos(3s)+1]}=$$
$$=\lim_{s\to 0}-\frac{\sin^2(3s)}{\sin^2(5s)[\cos(3s)+1]}.$$

Because $\lim_{s\to 0}[\cos(3s)+1]=\cos 0+1=2$ we have to compute the limit $\lim_{s\to 0}-\frac{1}{2}\times\frac{\sin^2(3s)}{\sin^2(5s)}$. We will do it with the help of our basic trigonometric limit $\lim_{u\to 0}\frac{\sin u}{u}=1$ as follows. $\lim_{s\to 0}-\frac{1}{2}\times\frac{\sin^2(3s)}{\sin^2(5s)}=\lim_{s\to 0}-\frac{1}{2}\times\frac{\sin^2(3s)}{(3s)^2}\times\frac{(5s)^2}{\sin^2(5s)}\times\frac{9}{25}=-\frac{9}{50}$.
(The second and the third factors in the computation above have limits equal to 1 because $\lim_{u\to 0}\frac{\sin^2 u}{u^2}=1^2=1$.)

Review of extra credit problems.

5. Prove using the precise definition of limit that

$\lim_{x\to 1}(2x^2-5x+2)=-1$. We have to prove that for any positive number ε we can find another positive number δ such that if $0<|x-1|<\delta$ then $|(2x^2-5x+2)-(-1)|<\varepsilon$. Let us fix a ε and search for an appropriate δ. If $0<|x-1|<\delta$ then
$|(2x^2-5x+2)-(-1)|=|2x^2-5x+3|=|(x-1)(2x-3)|=|x-1||2x-3|<\delta|2x-3|$.
To estimate the last expression we can assume that $\delta\le 1$ and therefore $|x-1|\le 1$ and $0\le x\le 2$. On the interval [0,2] the function $2x-3$ takes values from -3 to 1 whence $|2x-3|$ takes values from 0 to 3. In particular, if $\delta\le 1$

then $|2x-3|\leq 3$ and we have $|(2x^2-5x+2)-(-1)|<3\delta$. Finally we see that if $\delta<\min\left(1,\frac{\varepsilon}{3}\right)$ then $|(2x^2-5x+2)-(-1)|<\varepsilon$ and that proves our statement.

6. Find the limit $\lim\limits_{x\to 0}\frac{\sin x-\tan x}{x^3}$.

$\lim\limits_{x\to 0}\frac{\sin x-\tan x}{x^3}=\lim\limits_{x\to 0}\frac{\sin x-\frac{\sin x}{\cos x}}{x^3}=\lim\limits_{x\to 0}\frac{\sin x\cos x-\sin x}{x^3\cos x}$. Next, because $\lim\limits_{x\to 0}\cos x=\cos 0=1$ we have

$\lim\limits_{x\to 0}\frac{\sin x\cos x-\sin x}{x^3\cos x}=\lim\limits_{x\to 0}\frac{\sin x\cos x-\sin x}{x^3}=\lim\limits_{x\to 0}\frac{\sin x}{x}\times\frac{\cos x-1}{x^2}$. But $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$ and

$\lim\limits_{x\to 0}\frac{\cos x-1}{x^2}=\lim\limits_{x\to 0}\frac{(\cos x-1)(\cos x+1)}{x^2(\cos x+1)}=\lim\limits_{x\to 0}\frac{\cos^2 x-1}{2x^2}=\lim\limits_{x\to 0}-\frac{\sin^2 x}{2x^2}=-\frac{1}{2}$.

Therefore $\lim\limits_{x\to 0}\frac{\sin x-\tan x}{x^3}=-\frac{1}{2}$.