Math 171 **Review for the Final Exam**

1. Find the limits (4 points each)

   (a) $\lim\limits_{x\to\infty} \frac{\sqrt{4x^2-3}}{x}$;

   (b) $\lim\limits_{x\to\infty} (\frac{x-2}{x-1})^x$;

   (c) $\lim\limits_{x\to 1}(\frac{1}{\ln x} - \frac{1}{x-1})$;

   (d) $\lim\limits_{x\to 2} \frac{\sin(x-2)}{x^2-4}$.

   Solutions (a) The limit $\lim\limits_{x\to\infty} \frac{\sqrt{4x^2-3}}{x}$ represents an indeterminate form $\frac{\infty}{\infty}$. Therefore we can keep only the leading terms in the numerator and in the denominator of the fraction.
   $\lim\limits_{x\to\infty} \frac{\sqrt{4x^2-3}}{x} = \lim\limits_{x\to\infty} \frac{\sqrt{4x^2}}{x}$. Because x tends to positive infinity we have $\sqrt{4x^2} = 2x$ whence $\lim_{x\to\infty} \frac{\sqrt{4x^2-3}}{x} = \lim\limits_{x\to\infty} \frac{2x}{x} = 2$.

   (b) The limit $\lim\limits_{x\to\infty} (\frac{x-2}{x-1})^x$ represents an indeterminate form 1^∞. As always when we deal with such an indeterminate form we will first find the limit of the natural logarithm of our expression. We have $\ln(\frac{x-2}{x-1})^x = x \ln \frac{x-2}{x-1}$, and the limit $\lim\limits_{x\to\infty} x \ln \frac{x-2}{x-1}$ is an indeterminate form $0 \times \infty$ because the first factor tends to ∞ and the second one - to $\ln 1 = 0$. To apply the L'Hospital's rule we have to write the last limit as an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. In our case it is easier to write it in the form $\frac{0}{0}$, namely, $\lim\limits_{x\to\infty} x \ln \frac{x-2}{x-1} = \lim\limits_{x\to\infty} \frac{\ln \frac{x-2}{x-1}}{\frac{1}{x}}$. According to the L'Hospital's rule we will now differentiate the numerator and the denominator and look at the following limit (when differentiating the numerator we combine the chain rule and the quotient rule).

$$\lim_{x\to\infty} \frac{\frac{x-1}{x-2}\frac{1(x-1)-(x-2)1}{(x-1)^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{(x-2)(x-1)}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-x^2}{(x-2)(x-1)} = -1.$$

   Therefore $\lim\limits_{x\to\infty} (\frac{x-2}{x-1})^x = e^{-1} = \frac{1}{e}$.

(c) We have here an indeterminate form $\infty - \infty$. First we bring it to the form $\frac{0}{0}$; $\lim\limits_{x\to 1}(\frac{1}{\ln x} - \frac{1}{x-1}) = \lim\limits_{x\to 1} \frac{(x-1)-\ln x}{(x-1)\ln x}$.

Next we use the L'Hospital's rule to obtain the following limit

$$\lim_{x\to 1} \frac{1 - \frac{1}{x}}{\ln x + \frac{x-1}{x}}$$

.

It is still an indeterminate form $\frac{0}{0}$ so we apply the L'Hospital's rule once again and get the limit

$$\lim_{x\to 1} \frac{\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}.$$

(d)

$$\lim_{x\to 2} \frac{\sin(x-2)}{x^2 - 4}$$

.

This is an indeterminate form $\frac{0}{0}$ and we apply the L'Hospital's rule.

$$\lim_{x\to 2} \frac{\cos(x-2)}{2x} = \frac{1}{4}.$$

2. Find the derivatives of $y = f(x)$ with respect to x (4 points each)

   (a) $f(x) = \cos(\sin(x+1))$;

   (b) $f(x) = \frac{x}{\sqrt{1+x^2}}$;

   (c) $f(x) = \sqrt{x}\sin\sqrt{x}$;

   (d) $5y^2 + \sin y = x^2$.

Solutions.

(a) $f(x) = \cos(\sin(x+1))$. We apply the chain rule to get

$$f'(x) = -\sin(\sin(x+1))\cos(x+1).$$

(b) $f(x) = \frac{x}{\sqrt{1+x^2}}$. By the quotient rule combined with the chain rule we have

$$f'(x) = \frac{\sqrt{1+x^2} - x\frac{2x}{2\sqrt{1+x^2}}}{1+x^2}.$$

We simplify the last expression by multiplying both numerator and denominator by $\sqrt{1+x^2}$.

$$f'(x) = \frac{(1+x^2) - x^2}{(1+x^2)\sqrt{1+x^2}} = \frac{1}{(1+x^2)^{3/2}}.$$

(c) $f(x) = \sqrt{x}\sin\sqrt{x}$. By the product rule combined with the chain rule we have

$$f'(x) = \frac{1}{2\sqrt{x}}\sin\sqrt{x} + \sqrt{x}\cos\sqrt{x}\frac{1}{2\sqrt{x}} = \frac{1}{2}\Big(\frac{1}{\sqrt{x}}\sin\sqrt{x} + \cos\sqrt{x}\Big).$$

If we recall the formula

$$A\cos\theta + B\sin\theta = \sqrt{A^2+B^2}\sin(\theta + \arctan\frac{B}{A})$$

then we can also write our answer as

$$f'(x) = \frac{1+x}{2x}\sin(\sqrt{x} + \arctan\sqrt{x}).$$

(d) $5y^2 + \sin y = x^2$. We apply implicit differentiation to get

$$10y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 2x.$$

Solving it for $\frac{dy}{dx}$ we get

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}.$$

3. Find the integrals (5 points each)

(a) $\int \frac{x}{(3x^2+1)^2}\,dx$;

(b) $\int \tan^2 x \sec^2 x\,dx$;

(c) $\int\limits_0^1 (x^3 - 2x + 1)\,dx$;

(d) $\int\limits_{-1}^0 6t^2(t^3+1)^{10}\,dt$.

Solutions

(a)

$$\int \frac{x}{(3x^2+1)^2}\,dx$$

.

If we notice that x is proportional to the derivative of $3x^2+1$ we can use the following substitution; $u = 3x^2+1$. Then $\frac{du}{dx} = 6x$, $du = 6xdx$, and therefore $xdx = \frac{1}{6}du$. The integral becomes

$$\frac{1}{6}\int \frac{1}{u^2}\,du.$$

Applying the power rule we get

$$\frac{1}{6}\int \frac{1}{u^2}\,du = -\frac{1}{6u} + C = -\frac{1}{6(3x^2+1)} + C.$$

(b)

$$\int \tan^2 x \sec^2 x\,dx$$

.

Notice that $\sec^2 x$ equals to the derivative of $\tan x$ and therefore it is convenient to use the substitution $u = \tan x$. Then $du = \sec^2 x dx$ and our integral becomes

$$\int u^2\,du = \frac{u^3}{3} + C = \frac{\tan^3 x}{3} + C.$$

(c) We apply the power rule and the Newton - Leibnitz formula to get

$$\int_0^1 (x^3 - 2x + 1)\,dx = \left(\frac{x^4}{4} - x^2 + x\right)\Bigg|_0^1 = \frac{1}{4}.$$

(d)

$$\int_{-1}^{0} 6t^2(t^3+1)^{10}\,dt.$$

We use the substitution $u = t^3 + 1$. Then $du = 3t^2dt$ whence $6t^2dt = 2du$. If $t = -1$ then $u = 0$, and if $t = 0$ then $u = 1$; we will change the limits of integration accordingly.

$$\int_{-1}^{0} 6t^2(t^3+1)^{10}\,dt = \int_0^1 2u^{10}\,du = \frac{2u^{11}}{11}\Bigg|_0^1 = \frac{2}{11}.$$

4. Use a linear approximation to estimate the following values (5 points each)

   (a) $\sqrt[3]{63}$;

   (b) $\sin 31°$.

Solutions In both cases we use the formula for linear approximation

$$L(x) = f(a) + f'(a)(x - a)$$

where point a should satisfy two (informal) conditions

- a should be close to x
- the values of $f(a)$ and $f'(a)$ should be easy to compute.

(a) In this case we take $f(t) = \sqrt[3]{t}$ whence $f'(t) = \frac{1}{3\sqrt[3]{t^2}}$. We also take $a = 64$ and $x = 63$. Then $f(a) = 4$, $f'(a) = \frac{1}{48}$, and $x - a = -1$. The corresponding linear approximation to $\sqrt[3]{63}$ is

$$4 - \frac{1}{48} \approx 3.979$$

(b) Remember that we do not use the degree measure in calculus. In our case we take

$$f(t) = \sin t, a = 30^\circ = \frac{\pi}{6} \text{ and } x = 31^\circ = \frac{\pi}{6} + \frac{\pi}{180}.$$

Then

$$f(a) = \sin\frac{\pi}{6} = \frac{1}{2}, f'(a) = \cos a = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \text{ and } x - a = \frac{\pi}{180}.$$

Therefore the value of the linear approximation is

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\pi}{180} \approx 0.515.$$

5. Find the values of a and b, if the tangent line to $y = ax^2 - bx$ at $(-2, 5)$ has slope $m = 2$. (5 points)
Solution. Because $y(-2) = 5$ we have the following equation for a and b.

$$4a + 2b = 5$$

The derivative $\frac{dy}{dx}$ is $2ax - b$. Because $\frac{dy}{dx}(-2) = 2$ we have the second equation for a and b.

$$-4a - b = 2.$$

Adding the equations we get

$$b = 7$$

and plugging this value of b into the first equation we get $4a + 14 = 5$ whence

$$a = -\frac{9}{4}.$$

6. If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $h(t) = 80t - 16t^2$. What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down? (5 points)
Solution The velocity of the ball equals to the derivative of its height.

$$v(t) = \frac{dh}{dt} = 80 - 32t.$$

We can find the moments of time when the ball is 96 ft high by solving the quadratic equation

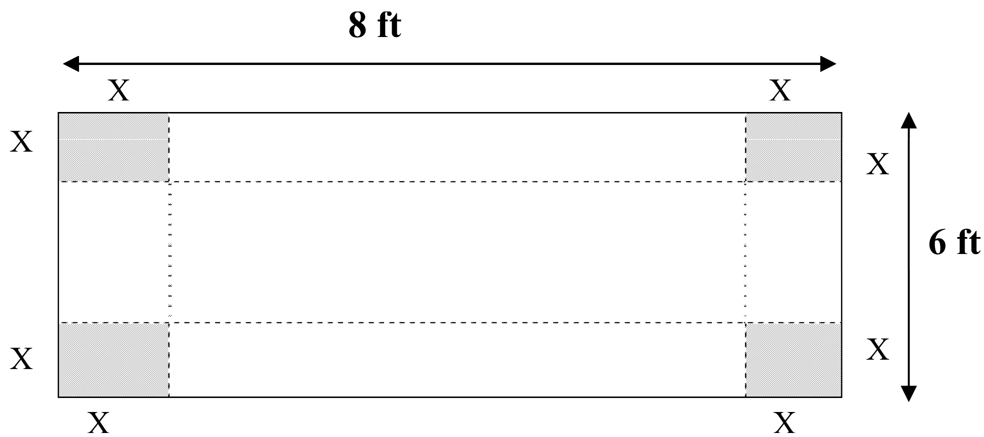
$$80t - 16t^2 = 96.$$

Dividing both parts by -16 and moving all terms to the left we get

$$t^2 - 5t + 6 = 0$$

whence $t = 2$ or $t = 3$. At the moment $t = 2$ the velocity of the ball is 16 ft/sec and the ball is moving up whilst at the moment $t = 3$ the velocity is -16 ft/sec and the ball is moving down.

7. An open box is to be made from a 6ft by 8ft rectangular piece of sheet metal by cutting out squares of equal size from the four corners and bending up the sides. Find the maximum value that the volume of the box can have.(10 points)
   Solution Let x be the side of each cut out square. Then the height of the box is x
   whilst its length is $8-2x$ and its width is $6-2x$ (See the picture below).


8 ft
X
X
X
X
6 ft
X
X
X
X

Notice that because these dimensions cannot be negative we have $0 \leq x \leq 3$. There fore we have to maximize the volume of the box

$$V(x) = x(8-2x)(6-2x) = 4x(4-x)(3-x) = 4x(x^2-7x+12) = 4(x^3-7x^2+12x)$$

on the interval $[0, 3]$. At the ends of the interval the function V takes value 0 so it will take the greatest value at a stationary point inside the interval. To find the stationary points we have to solve the equation

$$\frac{dV}{dx} = 4(3x^2 - 14x + 12) = 0.$$

The quadratic formula provides two solutions

$$x = \frac{14 \pm \sqrt{14^2 - 4 \times 3 \times 12}}{6} = \frac{14 \pm \sqrt{52}}{6} = \frac{7 \pm \sqrt{13}}{3}.$$

The sign + provides a solution which is greater than 3 and of no interest to us. The only stationary point inside the interval $[0, 3]$ is

$$x = \frac{7 - \sqrt{13}}{3} \approx 1.131.$$

You can easily check yourself that the greatest value of the volume is

$$V_{\max} = 4\frac{7 - \sqrt{13}}{3} \times \frac{5 + \sqrt{13}}{3} \times \frac{2 + \sqrt{13}}{3} = \frac{4(70 + 26\sqrt{13})}{27} = \frac{8(35 + 13\sqrt{13})}{27} \approx 25.767 ft^3.$$

8. Sketch the graph of $y = x^3 - 3x + 2$. Find x- and y- intercepts; plot the stationary points and the inflection points and determine the intervals where y is increasing and decreasing, concave up and concave down.(10 points)

Solution. To find the x-intercepts we have to solve the equation

$$x^3 - 3x + 2 = 0.$$

Notice that $x = 1$ is a solution of this equation and therefore $x - 1$ is a factor of $x^3 - 3x + 2$. We can factor $x^3 - 3x + 2$ using long division, grouping, or synthetic division. The table below shows synthetic division.

1	1	0	-3	2
		1	1	-2
	1	1	-2	0

Therefore $x^3 - 3x + 2 = (x-1)(x^2 + x - 2) = (x-1)^2(x+2)$ and the x-intercepts are $(-2, 0)$ and $(1, 0)$.

The y-intercept is $(0, 2)$.

The first derivative of y is

$$\frac{dy}{dx} = 3x^2 - 3 = 3(x-1)(x+1).$$

The function has two stationary points, -1 and 1 which divide the x-axis into the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. The next table shows the behavior of the function on these intervals.

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Sign of $\frac{dy}{dx}$	$+$	$-$	$+$
Behavior of y	Increases $\nearrow$	Decreases $\searrow$	Increases $\nearrow$

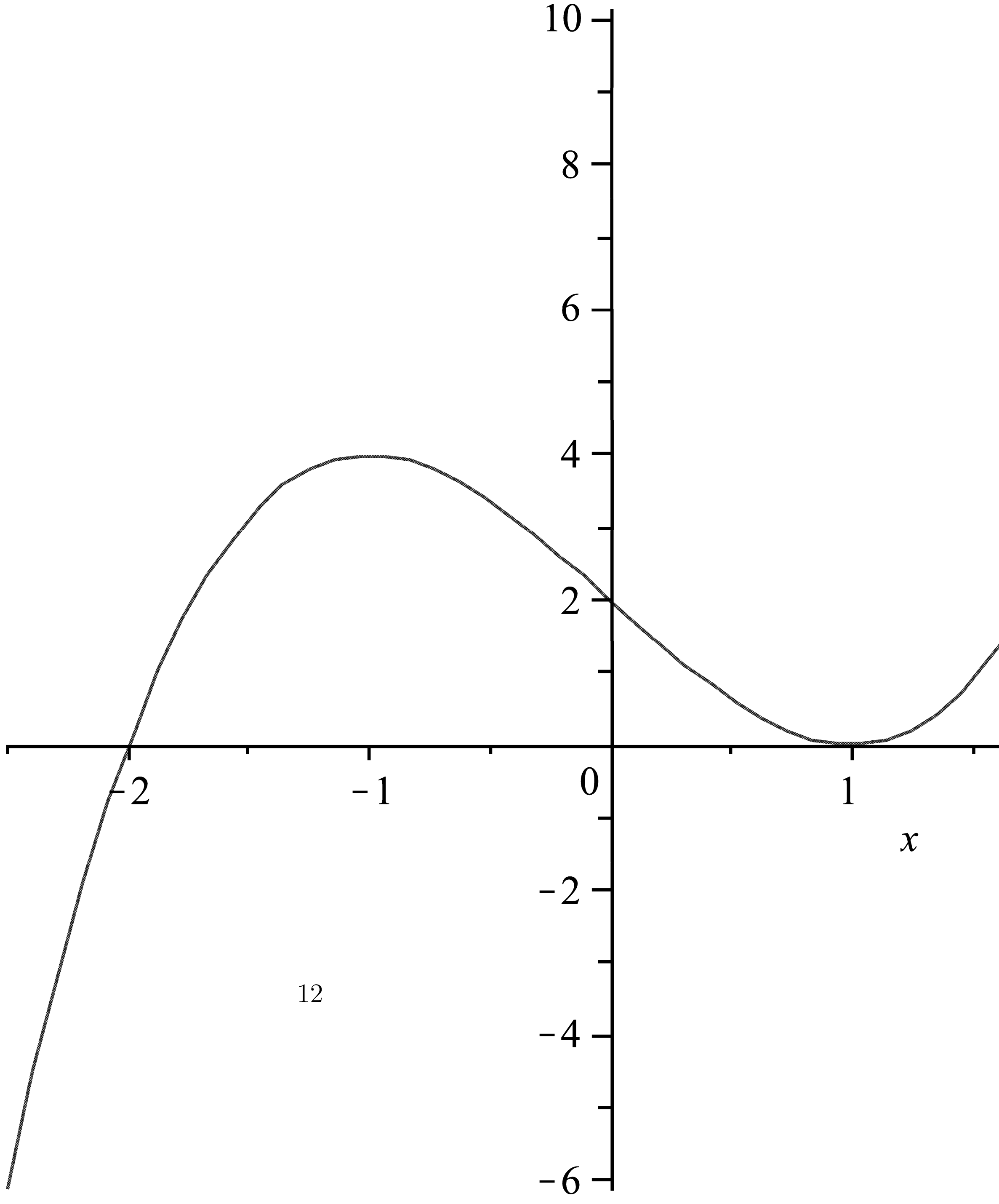
To find the inflection points of the cubic function y we compute its second derivative

$$\frac{d^2y}{dx^2} = 6x.$$

We see that y has only one inflection point 0. The information about concavity of y is contained in the next table.

Interval	$(-\infty, 0)$	$(0, \infty)$
Sign of $\frac{d^2y}{dx^2}$	-	$+$
Concavity	Down $\cap$	Up $\cup$

A computer generated graph of the function $y = x^3 - 3x + 2$ is shown below.


10
8
6
4
2
0
−2
−4
−6
−2
−1
1
x

9. Use the fundamental theorem of calculus:(4 points each)

(a) Find the derivative of the function

$$G(x) = \int_{x^2}^{\sin x} \sin t^2 \, dt;$$

Solution. The fundamental theorem of calculus in its generalized form tells us that

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t)\, dt = f(b(x)) \frac{db}{dx} - f(a(x)) \frac{da}{dx}.$$

Applying this formula when $a(x) = x^2$, $b(x) = \sin x$, and $f(t) = \sin t^2$ we get

$$\frac{dG}{dx} = \cos x \sin(\sin^2 x) - 2x \sin(x^4).$$

(b) Prove that the function

$$F(x) = \int_{2x}^{4x} \frac{2}{t}\, dt$$

is a constant on the interval $(-\infty, 0)$.

Solution. Look at two ways to solve the problem.

First way. We apply the fundamental theorem of calculus to see that

$$\frac{dF}{dx} = \frac{2}{4x}4 - \frac{2}{2x}2 = \frac{2}{x} - \frac{2}{x} \equiv 0.$$

Because the derivative of F is identically 0 the function F is a constant function.

Second way.

$$F(x) = \int_{2x}^{4x} \tfrac{2}{t}\, dt = 2\ln|t| \Big|_{2x}^{4x} =$$

$$= 2\ln|4x| - 2\ln|2x| = 2\ln 4 + 2\ln|x| - 2\ln 2 - 2\ln|x| =$$

$$= 2\ln 4 - 2\ln 2 = 2\ln 2.$$