**272 Differential equations** **Review 1**

1. A body of mass *m* is projected vertically upward with an initial velocity $v_0$ in a medium offering a resistance $k|v|$, where *k* is a constant. Assume that the gravitational attraction of the earth is constant.
(a) Find the velocity $v(t)$ of the body at any time.
(b) Use the result of part (a) to calculate the limit of $v(t)$ as $k \to 0$, that is, as the resistance approaches zero. Does this result agree with the velocity of a mass *m* projected upward with an initial velocity $v_0$ in a vacuum?
(c) Use the result of part (a) to calculate the limit of $v(t)$ as $m \to 0$, that is, as the mass approaches zero.

**Solution; (a)** the force applied to the body at any moment *t* is the sum of the gravitational force $F_1 = -mg$, where $g \approx 9.8 m/\sec^2$ ( the sign minus indicates that this force is directed down), and the force of resistance $F_2 = -kv$. Pay attention that when the velocity *v* is positive, the body moves up and the force of resistance is negative and directed down; on the other hand, when *v* is negative the body moves down, and the force of resistance is positive and directed up.

By the second law of Newton the acceleration $a(t)$ at moment *t* is equal to

$$a(t) = \frac{F(t)}{m} = \frac{F_1 + F_2}{m} = -g - \frac{kv(t)}{m}.$$

Therefore we have the following linear differential equation of the first order for the velocity *v*.

$$\frac{dv}{dt} + \frac{k}{m}v = -g.$$

To solve this equation we will follow the method on page 36. In notations on this page we have $p(t) = \frac{k}{m}$ and $g(t) = -g$. Formula (30) on page 36 yields

$$\mu(t) = \exp\int p(t)dt = \exp\int\left(\frac{k}{m}\right)dt = e^{(k/m)t}.$$

By formula (33) on the same page

$$v(t) = \frac{1}{\mu(t)}\left[\int_0^t \mu(s)g(s)ds + C\right] = e^{-(k/m)t}\left[-\int_0^t e^{(k/m)s} g ds + C\right] =$$

$$= e^{-(k/m)t}\left[\frac{mg}{k}(1 - e^{(k/m)t}) + C\right] = -\frac{mg}{k} + \left(\frac{mg}{k} + C\right)e^{-(k/m)t}.$$

The constant *C* can be found from the initial condition $v(0) = v_0$ whence $C = v_0$ and

$$v(t) = -\frac{mg}{k} + \left(\frac{mg}{k} + v_0\right)e^{-(k/m)t}.$$

**(b)** We have to find the limit of the above expression when $k \to 0$.

$$\lim_{k\to 0}\left(-\frac{mg}{k}+\left(\frac{mg}{k}+v_0\right)e^{-(k/m)t}\right)=\lim_{k\to 0} mg\,\frac{e^{-(k/m)t}-1}{k}+\lim_{k\to 0} v_0 e^{-(k/m)t}.$$

Clearly, $\lim_{k\to 0} v_0 e^{-(k/m)t}=v_0$.

By L'Hospital's rule $\lim_{k\to 0} mg\,\frac{e^{-(k/m)t}-1}{k}=\lim_{k\to 0} mg\,\frac{-\frac{t}{m}e^{-(k/m)t}}{1}=-gt$, whence

$$\lim_{k\to 0} v(t)=v_0-gt.$$

The last expression agrees completely with equations of motion in vacuum.

**(c)** $\lim_{m\to 0} v(t)=\lim_{m\to 0}-\frac{mg}{k}+\left(\frac{mg}{k}+v_0\right)e^{-(k/m)t}=0$.

2. A spring–mass system has a spring constant of $3N/m$. A mass of 2 kg is attached to the spring and the motion takes place in a viscous fluid that offers a resistance numerically equal to the magnitude of the instantaneous velocity. If the system is driven by an external force of $(3\cos 3t\ -\ 2\sin 3t)N$, solve the equation of motion and determine the steady-state response in the form $R(\cos\omega t-\delta)$.

**Solution;** the equation of motion is (compare with formula (1) on page 207)

$$mu''+\gamma u'+ku=F(t)$$

where $m$ is the mass, $\gamma$ is the damping coefficient, $k$ is the spring constant, and $F(t)$ is the external force. In our case $m=2,\gamma=1,k=3,$ and $F(t)=(3\cos 3t\ -\ 2\sin 3t)$. The equation of motion becomes

$$2u''+u'+3u=3\cos 3t-2\sin 3t.$$

First we solve the homogeneous equation

$$2u''+u'+3u=0.$$

The characteristic equation $2\lambda^2+\lambda+3=0$ has solutions $\lambda=-\frac{1}{4}\pm\frac{\sqrt{23}}{4}i$ and therefore the general solution of the homogeneous equation is

$$u_1(t)=e^{-(1/4)t}\left(C_1\cos\left(\frac{\sqrt{23}}{4}t\right)+C_2\sin\left(\frac{\sqrt{23}}{4}t\right)\right).$$

Next we look for a particular solution of the non-homogeneous equation $2u''+u'+3u=3\cos 3t-2\sin 3t$ in the form $u_2(t)=A\cos 3t+B\sin 3t$.

Then $u_2'(t)=-3A\sin 3t+3B\cos 3t$, $u_2''(t)=-9A\cos(3t)-9B\sin(3t)$, and $2u_2''(t)+u_2'(t)+3u_2(t)=(3B-15A)\cos 3t+(-3A-15B)\sin 3t$. Therefore

$$3B-15A=3$$

$$3A+15B=2$$

Solving this system we get $A=-\frac{1}{6},B=\frac{1}{6}$ and

$$u_2(t)=\frac{1}{6}\left(\sin 3t-\cos 3t\right)=-\frac{\sqrt{2}}{6}\cos(3t+\pi/4).$$

The general solution is $u(t) = u_1(t) + u_2(t)$ and because $u_1(t)$ is going to 0 exponentially when $t \to \infty$ the steady-state response equals to $u_2(t) = -\frac{\sqrt{2}}{6}\cos(3t + \pi/4)$.

3. Solve the equation $y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0$.

**Solution;** first we will solve the homogeneous equation

$$y'' + 4y' + 4y = 0.$$

The corresponding characteristic equation $\lambda^2 + 4\lambda + 4 = 0$ has solution -2 of multiplicity 2 and therefore the general solution of the homogeneous equation is $y = (A + Bt)e^{-2t}$.

Now we can apply Theorem 3.7.1 on page 189. We can take as two linearly independent solutions of the homogeneous equation the functions $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$. Then the Wronskian $W(y_1, y_2)(t)$ is computed as (see page 146)

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-4t}\begin{vmatrix} 1 & t \\ -2 & 1-2t \end{vmatrix} =$$

$$= e^{-4t}[1 - 2t - (-2t)] = e^{-4t}$$

Applying formula (28) on page 189 we see that a particular solution of our non-homogeneous equation can be found as

$$Y(t) = -e^{-2t}\int_1^t \frac{se^{-2s}s^{-2}e^{-2s}}{e^{-4s}}ds + te^{-2t}\int_1^t \frac{e^{-2s}s^{-2}e^{-2s}}{e^{-4s}}ds =$$

$$= -e^{-2t}\int_1^t \frac{1}{s}ds + te^{-2t}\int_1^t \frac{1}{s^2}ds = e^{-2t}(t - \ln t - 1).$$

The general solution can be written as

$$y = e^{-2t}(At + B - \ln t).$$